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# The development of nonlinear waves on the surface of a horizontally rotating thin liquid film

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We consider the axisymmetric thin liquid film formed on a horizontally spinning disk. The asymptotic structure of the steady film is obtained, after which a theory is developed to describe the evolution of localized disturbances imposed upon the steady film. It is shown that this can lead to the propagation of large gradients in the film. Moreover, it is found that under certain conditions the steady film can become unstable.

## 1. Introduction

The generation of thin, free-surface, liquid films is of practical importance to many processes arising in the field of chemical engineering. Particular examples are to be found in most heat and mass transfer and coating processes. One method of generating a thin-film fluid flow is to allow the fluid to run down a uniform plane which is inclined at an angle to the horizontal. This allows the formation of a steady flowing film of uniform thickness. Much experimental work has been devoted to the study of such films (see, for example, Kirkbride 1934, Friedman & Miller 1941, Grimley 1945, Kapitza & Kapitza 1949, Dukler & Bergelin 1952, Binnie 1959 and Tailby & Portalski 1962). For non-vertical inclinations, the uniform film is seen to be stable at sufficiently low flow rates, but there is an observed critical flow rate after which there is a tendency for the film to diverge from uniform conditions and develop into a periodic progressing wave. For vertical slopes, the uniform film is observed to be unstable at all flow rates.

The theoretical aspects of two-dimensional thin-film flows down uniform inclined planes have received much attention. A simple exact solution of the Navier–Stokes equations which satisfies all the boundary conditions is available to describe the uniform film flow. The temporal stability of the uniform flow when subject to small amplitude disturbances has been considered by Benjamin (1957). This analysis demonstrates that the stability of the uniform film depends primarily on the Reynolds number $R_f = Q/\nu$, where $Q$ is the volume flux in the film and $\nu$ is the kinematic viscosity of the fluid. In the 'thin-film' limit the exact condition for instability was obtained as $R_f > \frac{5}{6}\cot\theta$, with $\theta$ being the angle of inclination of the plane with the horizontal.

An alternative approach enabling the study of finite amplitude disturbances to the uniform film was given by Benney (1966). This reproduced the linearized stability condition of Benjamin (1957) and examined the possibility of weakly nonlinear equilibration of the film near conditions of neutral stability. The weakly nonlinear theory has also been considered by Gjevik (1970, 1971), who discusses the evolution of both spatially and temporally varying surface wave trains.

At low Reynolds numbers for which the uniform films are stable, permanent travelling waves have been discussed by Smith (1972). These waves represent transitional states which accommodate a change in the flow rate generating the film. Smith (1967, 1969) has also considered steady low-Reynolds-number films flowing down inclined planes containing small surface deformations.

Here we consider the theoretical aspects of a different method of generating thin-film fluid flows: the axisymmetric film generated on a horizontal disk by spinning. The film is generated by the drawing of fluid onto the spinning disk through a small aperture of height $a$ at the bottom of a central, cylindrical, reservoir of radius $l$. We consider the flow in the 'thin film' regime when $\epsilon = a/l \ll 1$, with both $a$ and $l$ small in comparison to the radius of the disk. The steady film is obtained, and is shown to consist of two regions. There is a thin region of lengthscale $O(\epsilon)$ next to the aperture in which the film adjusts rapidly from its inlet conditions. Thereafter the film thickness gradually falls with the flow being maintained primarily through a balance between centrifugal force and viscous stress.

An important dimensionless parameter governing the stability of the steady films is shown to be the modified film Reynolds number $\overline{Re} = \Omega^2 Q_T / 2\pi g \nu$ when $\Omega$ is the angular speed of the disk, $Q_T$ is the total volume flux generating the film, $g$ is the acceleration due to gravity and $\nu$ is the kinematic viscosity of the fluid. For $\overline{Re} \ll 1$ a theory is developed for unsteady film flows. For small-amplitude disturbances it is shown that the steady state is temporally stable according to a linearized theory. For disturbances of finite amplitude nonlinear effects become significant leading to the possible development of forward facing 'fronts' of thickness $O(\epsilon)$, and the conditions for the formation of such fronts are obtained. In the long time, these fronts decay and the film remains stable. The unsteady response of the film to changes in the driving volume flux is also considered. An increase in volume flux causes front formation, while a decrease does not.

Finally the temporal stability of the steady film is considered at larger values of the Reynolds number, $\overline{Re}$. It is found that the film can become unstable when $\overline{Re} = O(1)$. The criterion for instability is given by $\overline{Re} > \frac{5}{6}$, which is analogous to that obtained by Benjamin (1957) for thin films flowing down inclined planes. It is expected that when the Reynolds number is such that this criterion is satisfied the film will develop into a propagating 'wavy' form.

## 2. Equations of motion and boundary conditions

We consider the axisymmetric flow of a thin film of an incompressible viscous liquid on a rotating, horizontal disk. The liquid is injected onto the disk at a specified flow rate through a small gap of height $a$ at the bottom of a cylindrical reservoir of radius $l$ situated at the centre of the disk. With the origin $O$ at the centre of the disk, we use the cylindrical polar coordinates $(r, \theta, z)$ in a frame of reference rotating with the disk, where $r$ measures radial distance from the centre of the disk, $\theta$ is the angle from some fixed radial line in the disk and $z$ measures distance vertically upwards. The coordinates and the geometry of the situation are shown in figure 1. Within this rotating frame of reference the equations of motion are the Navier–Stokes equations with the inclusion of terms accounting for the centripetal and Coriolis forces (see, for example, Batchelor 1974). To complete the problem, boundary conditions must be specified on the free surface of the liquid, on the disk surface and at the inlet. The

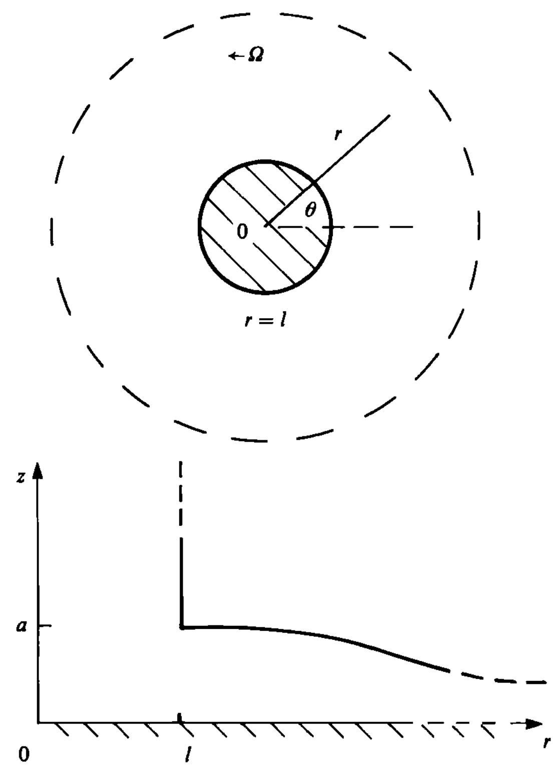


FIGURE 1. The coordinate system.

boundary conditions at the free surface, $z = D(r,t)$, are the usual kinematic and stress-free conditions, while we must have no slip of liquid on the disk surface, $z = 0$. At the inlet, $r = a$, we suppose that the liquid is injected through the gap as a plug flow with a volume flux $Q$ per unit length of perimeter (i.e. $Q_T = 2\pi l Q$). For unsteady flows further initial conditions must be specified.

We make the further assumption that the gap thickness of the reservoir, $a$, is much smaller than the radius $l$, which is in turn much smaller than the radius of the disk. Typical lengthscales for the flow in the horizontal and vertical directions are $l$ and $a$, respectively. With $a \ll l$, the leading-order balance in the radial momentum equation is between the viscous stress, the centripetal force and the pressure gradient. This results in a velocity scale $U_0$ in the radial direction and a pressure scale $P_0$ given by $U_0 = \Omega^2 l a^2/\nu$ and $P_0 = \rho\Omega^2 l^2$. The velocity scales in the azimuthal and vertical directions are then deduced from the azimuthal momentum and continuity equations as $V_0 = U_0 \Omega a^2/\nu$ and $W_0 = aU_0/l$ respectively. Finally, a timescale for the flow is taken as $T_0 = l/U_0$.

After using these scales to introduce appropriate dimensionless variables, the equations of motion become

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z}=0, \tag{1}$$

$$\epsilon Re\left\{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right\} = -\frac{\partial p}{\partial r}+r+2\epsilon Re\, v+\epsilon^2 Re^2\frac{v^2}{r}+\frac{\partial^2 u}{\partial z^2}+\epsilon^2\left\{\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right\}, \tag{2}$$

$$\epsilon Re\left\{\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}\right\}=-2u+\frac{\partial^2 v}{\partial z^2}+\epsilon^2\left\{\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right\}, \tag{3}$$

$$\epsilon^3 Re\left\{\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right\}=-\frac{\partial p}{\partial z}+\epsilon^2\frac{\partial^2 w}{\partial z^2}-\frac{\epsilon}{F}+\epsilon^4\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right), \tag{4}$$

where $u, v, w$ are the components of fluid velocity in the $r, \theta$ and $z$ directions, respectively, and $p$ is the fluid pressure. The three independent dimensionless parameters are $\epsilon = a/l$, the Reynolds number $Re = U_0 a/\nu$, and a Froude number $F = \Omega^2 l/g$. To complete the problem we have the following dimensionless boundary conditions,

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial r}-w=0, \tag{5}$$

$$\left.\begin{aligned} 2\epsilon^2\frac{\partial h}{\partial r}\left(\frac{\partial w}{\partial z}-\frac{\partial u}{\partial r}\right)+\left(\frac{\partial u}{\partial z}+\epsilon^2\frac{\partial w}{\partial r}\right)\left(1-\epsilon^2\left[\frac{\partial h}{\partial r}\right]^2\right)=0,\\ \frac{\partial v}{\partial z}-\epsilon^2 r\frac{\partial h}{\partial r}\frac{\partial}{\partial r}\left(\frac{v}{r}\right)=0,\end{aligned}\right\} \tag{6}$$

$$-p+2\epsilon^2\left[1+\epsilon^2\left(\frac{\partial h}{\partial r}\right)^2\right]^{-1}\left\{\epsilon\frac{\partial u}{\partial r}\left(\frac{\partial h}{\partial r}\right)^2+\frac{\partial w}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial h}{\partial r}-\epsilon^2\frac{\partial w}{\partial r}\frac{\partial h}{\partial r}\right\}$$
$$-\epsilon^3 T\frac{\partial^2 h}{\partial r^2}\left[1+\epsilon^2\left(\frac{\partial h}{\partial r}\right)^2\right]^{-\frac{1}{2}}=0 \quad \text{on } z=h(r,t), \quad r>1, \tag{7}$$

$$u=v=w=0 \quad \text{on } z=0, \quad r>1, \tag{8}$$

$$u=\sigma, \quad v=0, \quad w=0, \quad h=1 \quad \text{on } r=1 \quad \text{with } 0<z<1. \tag{9}$$

Here $h = D/a$, $T = \gamma/\Omega^2 l a^2$ (with $\gamma$ the coefficient of surface tension of the liquid) is a Weber number and $\sigma = Q/U_0 a$.

We consider solutions of (1)–(9) when $\epsilon \ll 1$ with $Re = o(1)$ and $F, \sigma, T = O(1)$ as $\epsilon \to 0$. It should be noted here that $\epsilon Re = E^{-2}$, where $E = \nu/\Omega a^2$ is the Ekman number for the flow. Thus we are considering flows with $E \gg 1$, which puts the restriction $\Omega \ll \nu/a^2$ on the speed of rotation of the disk.

## 3. The steady film

Here we obtain the form of the steady film as $\epsilon \to 0$. We look for solutions of (1)–(9) (with $\partial/\partial t \equiv 0$) as asymptotic expansions in the form

$$h = h_0(r)+\epsilon h_1(r)+\dots \quad \text{as } \epsilon\to 0 \quad \text{with } r=O(1), \tag{10a}$$

with similar expansions for $u$, $v$, $w$ and $p$.

On substituting the expansions into (1)–(4) and boundary conditions (5)–(9) we obtain at leading order the following equations:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_0)+\frac{\partial w_0}{\partial z}=0, \tag{10b}$$

$$\frac{\partial^2 u_0}{\partial z^2}+r-\frac{\partial p_0}{\partial r}=0, \tag{11}$$

$$\frac{\partial^2 v_0}{\partial z^2} - 2u_0 = 0, \tag{12}$$

$$\frac{\partial p_0}{\partial z} = 0, \tag{13}$$

which are to be solved together with the leading-order boundary conditions

$$u_0 \frac{\partial h_0}{\partial r} - w_0 = 0, \tag{14}$$

$$\frac{\partial u_0}{\partial z} = 0, \tag{15}$$

$$p_0 = 0, \tag{16}$$

$$\frac{\partial v_0}{\partial z} = 0, \tag{17}$$

on $z = h_0(r)$ with $r > 1$;

$$u_0 = v_0 = w_0 = 0 \quad \text{on } z = 0 \quad \text{with } r > 1; \tag{18}$$

$$u_0 = \sigma, \quad v_0 = w_0 = 0, \quad h_0 = 1 \quad \text{at } r = 1, \quad 0 < z < 1. \tag{19}$$

The solutions of equations (10*b*)–(13) which satisfy conditions (15)–(18) are readily obtained as

$$\left.\begin{aligned} p_0(r,z) &\equiv 0, \\ u_0(r,z) &= \tfrac{1}{2} rz[2h_0(r) - z], \\ v_0(r,z) &= \tfrac{1}{12} rz[4z^2 h_0(r) - z^3 - 8h_0^3(r)] \\ w_0(r,z) &= \tfrac{1}{3}z^3 - \tfrac{1}{2}z^2 r \frac{\mathrm{d}h_0}{\mathrm{d}r} - z^2 h_0(r). \end{aligned}\right\} \tag{20}$$

Substitution from (20) into condition (14) now provides an ordinary differential equation for $h_0(r)$, namely,

$$\frac{\mathrm{d}h_0}{\mathrm{d}r} + \frac{2}{3r} h_0 = 0. \tag{21}$$

The general solution of (21) is given by

$$h_0(r) = Ar^{-\frac{2}{3}}, \tag{22}$$

where $A$ is a constant. It remains to satisfy the conditions (19) at $r = 1$. From (20) and (22) it is clear that the constant $A$ cannot be chosen to satisfy all the conditions (19). This problem has arisen because, at leading order, the highest derivatives with respect to $r$ occurring in the full equations (2)–(4) have been neglected. We conclude that expansions of the form (10*a*) become non-uniform as $r \to 1$. To obtain expansions valid when $r \sim 1$ we introduce an inner region, with the expansions (10*a*) now viewed as outer expansions valid when $r$ is $O(1)$. The unknown constant $A$ arising in expression (22) is then determined by matching the outer expansions (10*a*) with expansions valid in the inner region.

In the inner region, $r = 1 + o(1)$ and $z = O(1)$ as $\epsilon \to 0$, and we must retain at leading order all the highest derivatives with respect to $r$ in equations (2)–(4) to enable all the conditions at $r = 1$ to be satisfied. The appropriate scalings for the inner region are then readily obtained as $r = 1+O(\epsilon)$, $h, u, v, z = O(1)$, $w = O(\epsilon^{-1})$ and $p = O(\epsilon)$. Thus we introduce the scaled inner variables $\bar{r}$, $\bar{w}$ and $\bar{p}$ defined by

$$\bar{r} = \frac{(r-1)}{\epsilon}, \quad \bar{w} = \epsilon w, \quad \bar{p} = \epsilon^{-1} p.$$

After writing the full equations and boundary conditions (1)–(9) in terms of these inner variables we look for solutions in the inner region as asymptotic expansions in the form

$$h = \bar{h}_0(\bar{r}) + \dots \quad \text{as } \epsilon, Re \to 0, \tag{23}$$

together with similar expansions for $u$, $v$, $\bar{w}$ and $\bar{p}$.

At leading-order (1) becomes

$$\frac{\partial \bar{u}_0}{\partial \bar{r}} + \frac{\partial \bar{w}_0}{\partial z} = 0, \tag{24}$$

which can be integrated once to give

$$\frac{\mathrm{d}}{\mathrm{d}\bar{r}} \int_0^{\bar{h}_0} \bar{u}_0 \,\mathrm{d}z - \bar{u}_0(\bar{r}, \bar{h}_0) \frac{\mathrm{d}\bar{h}_0}{\mathrm{d}\bar{r}} + \bar{w}_0 = 0. \tag{25}$$

Also, boundary condition (5) is, at leading order,

$$\bar{u}_0 \frac{\mathrm{d}\bar{h}_0}{\mathrm{d}\bar{r}} - \bar{w}_0 = 0, \tag{26}$$

and when (26) is substituted into (25) and the condition that $\bar{u}_0 = \sigma$, $\bar{w}_0 = 0$ at $\bar{r} = 1$ is applied, we obtain

$$\int_0^{\bar{h}_0} \bar{u}_0 \,\mathrm{d}z = \sigma. \tag{27}$$

We then evaluate (27) as $\bar{r} \to \infty$, using the matching condition, from (20), that $\bar{h}_0 \to A$, $\bar{u}_0 \to \frac{1}{2}z(2A-z)$ as $\bar{r} \to \infty$. A simple integration then gives the constant $A$ as

$$A = (3\sigma)^{\frac{1}{3}}. \tag{28}$$

The main purpose of introducing the inner region was to enable the determination of the unknown constant arising in the outer solutions (20), (22). This has now been achieved. It should be noted that the full problem in the inner region is a free-surface Stokes flow which can be reduced to the solution of a biharmonic equation by introducing a stream function. Due to the complexity of the nonlinear free-surface conditions, analytical solutions to this type of problem are not available, although the application of boundary-integral techniques to obtain numerical solutions for plane, free-surface Stokes flows has been considered by Kelmanson & Ingham (1984).

Thus for $\epsilon \ll 1$, the asymptotic structure of the film has two regions. In the thin inner region, of radial lengthscale $O(\epsilon)$, the film suffers rapid adjustments from its inlet conditions. The film thickness changes from 1 to $(3\sigma)^{\frac{1}{3}}$, the radial velocity at the free surface changes from $\sigma$ to $\frac{1}{2}(3\sigma)^{\frac{2}{3}}$ and the azimuthal velocity at the free surface changes from zero to $-\frac{5}{4}(3\sigma^4)^{\frac{1}{3}}$. After this rapid adjustment the film develops when $r$ is $O(1)$ according to the outer solutions (20) and (22). As $\epsilon \to 0$ a uniform approximation to the film can therefore be obtained from the leading-order outer solutions (20) and (22) together with the following discontinuity at $r = 1$,

$$[h] = (3\sigma)^{\frac{1}{3}} - 1, \tag{29}$$

to replace the boundary conditions (19). From (29) we see that the behaviour of the film close to the inlet depends on the parameter $\sigma$. With $\sigma > \frac{1}{3}$ there is a step up in film thickness at the inlet, and with $\sigma < \frac{1}{3}$ a step down. If $\sigma = \frac{1}{3}$, i.e. $Q_T/2\pi l = \Omega^2 l a^3/3\nu$, the flow rate at the inlet is the same as the flow required for a balance between centripetal and viscous forces for a film of thickness $a$ at radius $l$, and hence the film emerges from the inlet with this thickness without an initial adjustment of height in the inner region.

It is also important to note that the theory does not necessarily require $Re = o(1)$. The outer solutions remain the same at leading order when $Re$ is $O(1)$. The only modification for $Re$ of $O(1)$ is in the inner region, where the leading-order problem becomes a plane, free-surface flow governed by the full Navier–Stokes equations, rather than a Stokes problem.

## 4. The asymptotic structure of the unsteady film

We now consider the development of disturbances imposed upon the steady film discussed in the previous section. The disturbance can be of two types: a local disturbance imposed upon the film away from the inlet together with possible imposed flux changes at the inlet. Flux changes at the inlet are accommodated by setting $\sigma = \sigma(t)$ in the boundary condition (9), while the effects on the film of a local disturbance are considered by imposing the following initial conditions,

$$u(t=0) = U(r,z), \tag{30a}$$

$$v(t=0) = V(r,z), \tag{30b}$$

$$h(t=0) = H(r). \tag{30c}$$

Here $U$, $V$ and $H$ are bounded functions of $r$ and $z$ which satisfy the boundary conditions (9) at $r=1$. The initial form of $w$ cannot be prescribed independently of conditions (30) due to the continuity equation (1). An integration of this equation at $t=0$ together with (30) leads to

$$w(t=0) = -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\int_0^z U(r,s)\,\mathrm{d}s\right].$$

As $\epsilon \to 0$, the asymptotic structure of the solution of equations (1)–(4) subject to boundary conditions (5)–(9) and initial conditions (30) is similar to that of the steady film. When $t = O(1)$, two regions are required to obtain a uniform approximation to the film in $r > 1$, namely

*region* I: $r, z, t = O(1)$,
*region* II: $r = 1 + O(\epsilon)$, $z, t = O(1)$.

In both the above regions we lose at leading order the time derivatives of $u$, $v$ and $w$ in equations (2)–(4). Consequently the expansions obtained in I and II are unable to satisfy the conditions (30) at $t=0$. Thus we must conclude that the expansions in both I and II become non-uniform as $t \to 0$. Uniform expansions for $t \ll 1$ are obtained by introducing the additional regions III and IV with the following scalings,

*region* III: $r, z = O(1)$, $t = O(\epsilon Re)$,
*region* IV: $r = 1 + O(\epsilon)$, $z = O(1)$, $t = O(\epsilon Re)$.

The asymptotic matching of regions II–IV with region I then determines uniquely the solution in this main region.

We begin in region I by looking for a solution of equations (1)–(4) as asymptotic expansions in the form

$$h = h_0(r,t) + \epsilon h_1(r,t) + \dots, \tag{31}$$

with similar expansions for $u$, $v$, $w$ and $p$. On substituting from (31) into (1)–(4) and expanding we again obtain the leading-order equations as (10$b$)–(13). As noted earlier, the solution of (10$b$)–(13) will not be able to accommodate all the conditions (30) and (9) imposed at $t = 0$ and $r = 1$ respectively. Consequently we attempt to apply only conditions (5)–(8), which become at leading order

$$\frac{\partial h_0}{\partial t} + u_0 \frac{\partial h_0}{\partial r} - w_0 = 0 \quad \text{on } z = h_0, \tag{32$a$}$$

$$\left.\begin{aligned} \frac{\partial u_0}{\partial z} &= 0 \\ \frac{\partial v_0}{\partial z} &= 0 \\ p_0 &= 0 \end{aligned}\right\} \quad \text{on } z = h_0, \tag{32$b$}$$

$$u_0 = v_0 = w_0 \quad \text{on } z = 0. \tag{33}$$

The solution of equations (10$b$)–(13) which satisfies conditions (32$b$) and (33) is the same as that obtained for the steady film given in (20) except that now $h_0 = h_0(r,t)$ is also a function of $t$. On substituting for $u_0$ and $w_0$ from (20) into condition (32$a$) we obtain the equation determining $h_0$, namely,

$$\frac{\partial h_0}{\partial t} + r h_0^2 \frac{\partial h_0}{\partial r} = -\tfrac{2}{3} h_0^3. \tag{34}$$

It is now clear that $u_0, v_0, w_0$ as given by (20) with $h_0$ determined through (34) cannot satisfy all of the conditions (30) and (9) imposed at $t = 0$ and $r = 1$. To obtain the appropriate conditions to be satisfied by the solution of (34), and hence complete the leading-order problem in region I, we must introduce the further regions II–IV.

We consider first region II which is introduced because of the non-uniformity that arises in I as $r \to 1$ when $t = O(1)$. The scalings in this region are the same as those of the inner region for the steady film. Thus we introduce the scaled variables $\bar{r}, \bar{w}$, $\bar{p}$ as defined previously. The conditions to be applied in this region are (5)–(9), together with matching to I as $\bar{r} \to \infty, t = O(1)$ and IV as $t \to 0, \bar{r} = O(1)$. Equations (1)–(4) together with conditions (5)–(9) are written in terms of $\bar{r}$, $\bar{w}$ and $\bar{p}$, after which we look for a solution in II of the form

$$h = \bar{h}_0(\bar{r}, t) + \dots \quad \text{as } \epsilon, Re \to 0, \tag{35}$$

with similar expansions for $u$, $v$, $\bar{w}$ and $\bar{p}$.

At leading order we find that the problem in II which determines $\bar{u}_0$, $\bar{v}_0$, $\bar{w}_0$, $\bar{p}_0$ and $\bar{h}_0$ is the same as that obtained for the inner region of the steady film (with $\sigma = \sigma(t)$).

As with the steady film the leading-order problem in II is a Stokes-type problem which is analytically intractable. However, it is the matching to I which is of primary importance, as this determines the boundary conditions to be applied to the solution of (34) at $r = 1$. We are again fortunate in being able to obtain this condition without obtaining a full solution to the Stokes problem. We first match expansions (35) in II as $\bar{r} \to \infty$ with expansions (31) in I as $r \to 1$. This, on using (20), gives the following far-field conditions for the Stokes problem,

$$\bar{h}_0(\bar{r}, t) \to h_0(1, t), \tag{36a}$$

$$\bar{u}_0(\bar{r}, t) \to \tfrac{1}{2}z[2h_0(1, t) - z], \tag{36b}$$

$$\left.\begin{aligned} \bar{v}_0(\bar{r}, t) &\to \tfrac{1}{12}z[4h_0(1, t)\, z^2 - z^3 - 8h_0^3(1, t)], \\ \bar{w}_0(\bar{r}, t) &\to 0, \end{aligned}\right\} \tag{36c}$$

as $\bar{r} \to \infty$. In the same way as for the steady film, we can also show from (24), (25) and (26) that

$$\int_0^{\bar{h}_0(\bar{r}, t)} \bar{u}_0(\bar{r}, z, t)\, \mathrm{d}z = \sigma(t), \tag{37}$$

throughout region II. Finally we evaluate (37) as $\bar{r} \to \infty$ using (36$a, b$). On performing the integration this determines the required boundary condition for I as

$$h_0(1, t) = [3\sigma(t)]^{\frac{1}{3}}. \tag{38}$$

To proceed further in region II a numerical solution of the Stokes problem must be considered. We do not pursue this here, since our primary objective has already been achieved in obtaining condition (38).

The non-uniformity arising in region I as $t \to 0$ with $r = O(1)$ leads to the inability of the solution in I to satisfy all of the initial conditions (30). To overcome this, we must introduce region III with timescale $t = O(\epsilon Re)$. This scale is chosen to retain at leading order the appropriate time derivatives in equations (1)–(4) to enable all of (30) to be satisfied. The asymptotic matching of the expansions in I and III then determines the condition to be satisfied by the solution of (34) when $t = 0$. We introduce the scaled time $\hat{t} = t/\epsilon Re$ so that $\hat{t} = O(1)$ in III. Equations (1)–(4) are now written in terms of $\hat{t}$ and we look for solutions in III as asymptotic expansions of the form,

$$h = \hat{h}_0(r, z, \hat{t}) + \epsilon \hat{h}_1(r, z, \hat{t}) + \dots \quad \text{as } \epsilon \to 0, \tag{39}$$

with similar expansions for $u$, $v$, $w$ and $p$.

The leading-order problem is readily solved, to give

$$\hat{h}_0(r, \hat{t}) = H(r), \tag{40a}$$

$$\hat{p}_0(r, z, \hat{t}) \equiv 0, \tag{40b}$$

$$\hat{u}_0(r, z, \hat{t}) = \tfrac{1}{2}rz(2H(r) - z) + \sum_{n=0}^{\infty} B_n(r) \exp\left[-\frac{(n+\frac{1}{2})^2\pi^2}{H(r)^2}\hat{t}\right]\psi_n(z), \tag{40c}$$

$$\hat{v}_0(r, z, \hat{t}) = \tfrac{1}{12}rz(4H(r)\, z^2 - z^3 - 8H(r)^3) + \sum_{n=0}^{\infty} [A_n(r) - 2\hat{t}B_n(r)] \exp\left[-\frac{(n+\frac{1}{2})^2\pi^2}{H(r)^2}\hat{t}\right]\psi_n(z), \tag{40d}$$

$$\hat{w}_0(r, z, \hat{t}) = \frac{1}{r}\frac{\partial}{\partial r} r\left[\int_0^z \hat{u}_0(r, s, \hat{t})\, \mathrm{d}s\right]. \tag{40e}$$

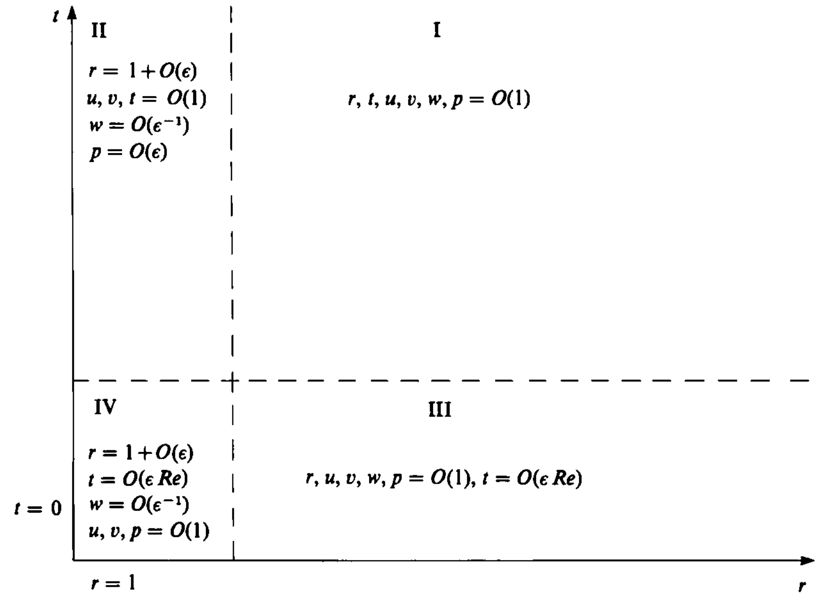


FIGURE 2. The asymptotic regions I–IV in the $(r, t)$-plane.

Here $\psi_n$ are the orthonormal eigenfunctions

$$\psi_n(z) = \frac{\sqrt{2}}{H^{\frac{1}{2}}} \sin \frac{(n+\frac{1}{2})\pi z}{H},$$

and the Fourier coefficients $A_n$ and $B_n$ are given by

$$A_n(r) = \int_0^{H(r)} [V(r,z) - \tfrac{1}{12}rz(4Hz^2 - z^3 - 8H^3)]\, \psi_n(z)\, \mathrm{d}z,$$

$$B_n(r) = \int_0^{H(r)} [U(r,z) - \tfrac{1}{2}rz(2H(r) - z)]\, \psi_n(z)\, \mathrm{d}z.$$

We now match expansions (39) as $\hat{t} \to \infty$ with expansions (31) as $t \to 0$. From (20) and (40*b–e*) it is clear that the matching of $u$, $v$, $w$ and $p$ is automatic. The matching of $h$ then leads to

$$h_0(r, 0) = H(r), \tag{41}$$

which provides the appropriate initial condition to be satisfied by the solution of (34). The solution in region I is now uniquely determined through (34) together with conditions (38) and (41). Before studying the solution of this problem, for completeness we first consider the remainder of the asymptotic structure of the film.

The structure is completed by the introduction of region IV when $t \ll 1$ and $(r-1) \ll 1$. This accommodates the non-uniformities in regions II and III as $t \to 0$ and $r \to 1$ respectively. The appropriate scalings for this region are found to be

$$r = 1 + O(\epsilon), \quad t = O(\epsilon Re), \quad p = O(\epsilon), \quad u, v = O(1), \quad w = O(\epsilon^{-1}).$$

At leading order in IV we obtain a time-dependent Stokes problem subject to boundary conditions on $\bar{r} = 0, z = 0, z = h$, initial conditions at $t = 0$, together with matching conditions with II as $\hat{t}\to\infty$ ($\bar{r}=O(1)$) and III as $\bar{r}\to\infty$ ($\hat{t}=O(1)$). Further details of this region are not pursued as it is passive, i.e. information from I–III is passed into IV through the matching conditions.

Again it should be noted that the asymptotic structure of this section holds when $Re=O(1)$, with the leading-order problems in I and III being unchanged, while those in II and IV become full Navier–Stokes rather than Stokes problems. The asymptotic regions I–IV are illustrated schematically in the $(r,t)$-plane in figure 2.

In the remainder of the paper attention is restricted to the solution of the problem in region I.

## 5. The solution in region I

### 5.1. *The initial-boundary-value problem*

In this section we obtain the solution, $h_0(r,t)$, of (34) subject to the initial and boundary conditions (41) and (38). It is assumed that both $\sigma(t)\,(t\geqslant 0)$ and $H(r)$ $(r\geqslant 1)$ are positive, bounded functions with $H(1)=[3\sigma(0)]^{\frac{1}{3}}$, $H(r)\to 0$ as $r\to\infty$. It is convenient first to make the transformations

$$x=\log r,\quad \psi=h_0^2. \tag{42}$$

In terms of $x$ and $\psi$, (34) becomes

$$\frac{\partial\psi}{\partial t}+\psi\frac{\partial\psi}{\partial x}=-\tfrac{4}{3}\psi^2, \tag{43}$$

whilst conditions (41) and (38) may be written as

$$\psi(x,0)=G^2(x)\quad(x\geqslant 0), \tag{44a}$$

$$\psi(0,t)=A^2(t)\quad(t\geqslant 0), \tag{44b}$$

where $G(x)\equiv H(\mathrm{e}^x)$ and $A(t)\equiv[3\sigma(t)]^{\frac{1}{3}}$. The solution of (43) subject to (44) can be obtained in implicit form via the method of characteristics.

We consider the curve in $x,t\geqslant 0$ defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\psi(x(t),t). \tag{45}$$

Upon this curve (43) reduces to the ordinary differential equation

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}=-\tfrac{4}{3}\psi^2. \tag{46}$$

Since $\psi>0$ for all $x,t\geqslant 0$ we note from (45) that the characteristic curves, $x(t)$ are monotone increasing functions of $t$. For a characteristic curve entering $x,t>0$ via the $x$-axis, (45) and (46) are solved subject to $x(0)=\xi$, $\psi(\xi,0)=G^2(\xi)$, which leads to the following implicit form of the solution:

$$\psi=[\tfrac{4}{3}t+G^{-2}(\xi)]^{-1}, \tag{47a}$$

on the curves

$$x^{+}(t)=\xi+\tfrac{3}{4}\log[\tfrac{4}{3}G^2(\xi)\,t+1], \tag{47b}$$

for all positive values of the parameter $\xi$. The solution is completed by considering the characteristic curves entering $x, t > 0$ via the $t$-axis. Equations (45) and (46) must be integrated subject to $x(\tau) = 0$, $\psi(0, \tau) = A^2(\tau)$, after which we obtain

$$\psi = [\tfrac{4}{3}(t-\tau) + A(\tau)^{-2}]^{-1}, \tag{48a}$$

on the curves
$$x^-(t) = \tfrac{3}{4}\log[\tfrac{4}{3}A^2(\tau)(t-\tau)+1], \tag{48b}$$

for all positive values of the parameter $\tau$.

The solution in $x, t > 0$, defined implicitly in (47) and (48), remains single valued provided no characteristic curves (47$b$) and (48$b$) intersect within this region. We consider this case first.

5.1.1. *No characteristic intersection*

The solution remains single-valued for all $t > 0$, with the characteristic curve $x_c(t) = \frac{3}{4}\log[\frac{4}{3}G^2(0)\,t+1]$ forming a dividing curve between information propagating due to boundary and initial conditions. For $x < x_c(t)$ the solution is determined by the boundary condition through (48), while for $x > x_c(t)$ the solution is determined by the initial condition through (47). In terms of the radial variable, this transition point occurs at

$$r_c(t) = (1+\tfrac{4}{3}G^2(0)\,t)^{\frac{3}{4}},$$

and we see that the effects of the initial conditions are 'swept away' with speed $\dot{r}_c(t) = G^2(0)[1+\frac{4}{3}G^2(0)\,t]^{-\frac{1}{4}}$.

5.1.2. *Characteristic intersection*

If either of the family of characteristic curves defined by (47$b$) and (48$b$) have intersections in $x, t > 0$, then at each such point the solution is multiple valued. We now examine the conditions under which characteristic intersection and hence multiple-valued solutions can occur. We assume for the moment that $A(t)$ and $H(r)$ are at least once differentiable. Now suppose that characteristic intersection does occur and let $t_B$ be the minimum value of $t$ at which intersection occurs (i.e. for $0 \leqslant t < t_B$ the solution is single-valued). Since the characteristic intersection at $t = t_B$ is the first it is easily deduced that it must occur between neighbouring characteristics. A characteristic from the family (47$b$) intersects its neighbour at $t = t_I^+(\xi)$ defined by $\mathrm{d}x^+/\mathrm{d}\xi\,|\,(t = t_I^+) = 0$. After differentiating (47$b$) we obtain

$$t_I^+(\xi) = \frac{-1}{2G(\xi)[G'(\xi)+\frac{2}{3}G(\xi)]} \quad (\xi \geqslant 0). \tag{49}$$

Thus, from (49), the set of characteristics (47$b$) intersect in $x, t > 0$ if and only if there exists a value of $\xi > 0$, say $\xi_0$, at which

$$[G'(\xi_0)+\tfrac{2}{3}G(\xi_0)] < 0. \tag{50}$$

When (50) is satisfied, the time at which intersection first occurs on this set of characteristics is given by

$$t_B^+ = \frac{1}{\{-2G(\xi)[G'(\xi)+\frac{2}{3}G(\xi)]\}_{\max_{\xi\geqslant 0}}}. \tag{51}$$

Similarly, neighbouring characteristics from the family (48$b$) intersect at $t = t_I^-(\tau)$ defined by $\mathrm{d}x^-/\mathrm{d}\tau(t = t_I^-) = 0$, which leads to

$$t_I^-(\tau) = \tau + \frac{A(\tau)}{2A'(\tau)} \quad (\tau \geqslant 0). \tag{52}$$

Hence, using (48*b*) and (52), characteristics from this family intersect in $x, t > 0$ if and only if there exists a value of $\tau > 0$, say $\tau_0$, at which

$$A'(\tau_0) > 0. \tag{53}$$

When (53) is satisfied the minimum intersection time for the characteristic family (48*b*) is given by

$$t_B^- = \left\{\tau + \frac{A(\tau)}{2A'(\tau)}\right\}_{\min_{\tau \in S}}, \tag{54}$$

where $S = \{\tau : A'(\tau) > 0\}$. After writing (50) in terms of $H$, the above results are most easily summarized in the following:

*Result* (*i*) The solution of (34) in $(r-1)$, $t > 0$ subject to the initial and boundary conditions (41) and (38) becomes multiple valued in $t > t_B$ if and only if there exists an $r_0 > 1$ such that

$$\left.\frac{\partial H}{\partial r}\right|_{r_0} + \frac{2}{3r_0} H(r_0) < 0, \tag{55}$$

or a $t_0 > 0$ such that

$$\left.\frac{dA}{dt}\right|_{t_0} > 0. \tag{56}$$

When only (55) is satisfied $t_B = t_B^+$, when only (56) is satisfied $t_B = t_B^-$; when both are satisfied $t_B = \min(t_B^+, t_B^-)$.

### 5.2. *Discontinuous solutions and discontinuity structure*

The above result establishes necessary and sufficient conditions for the appearance in finite time of multiple-valued solutions in region I. Since we expect that the solution of the full equations (1)–(9) will remain single valued in $(r-1), t > 0$ for any $\epsilon > 0$, the appearance of multiple-valued solutions of (34) suggests that the approximations made in I to arrive at (34) have become non-uniform in some region of the $(r, t)$-plane. This becomes clear when we observe that as $t \to t_B$ from below, $h_0(r, t)$ acquires a steep gradient in the vicinity of the point $r_B$ at which it first becomes multiple valued. When $t$ passes through $t_B$, $\partial h_0/\partial r \,|\, r_B$ becomes infinite, after which $h_0(r, t)$ 'folds' over, creating the initial multiple-valued region. By this stage, $h_0(r, t)$ will no longer provide a uniform approximation to the full solution in the neighbourhood of $(r_B, t_B)$, since in arriving at (34) we neglected the highest derivatives in $r$ from the full equations (1)–(9). This is no longer justifiable in the vicinity of $(r_B, t_B)$, where derivatives in $r$ have become large.

We now demonstrate that a uniform approximation in I can be recovered by taking the multiple-valued solution, $h_0(r, t)$ of (34) and replacing the multiple-valued region by an appropriate jump discontinuity in $h_0(r, t)$. This discontinuous solution provides a uniform approximation everywhere except in the neighbourhood of the discontinuity. An appropriate scaling of the full equations then provides a leading-order theory uniform in the vicinity of the discontinuity. The asymptotic matching of the solution in this 'discontinuity structure' region to the outer solution $h_0(r, t)$ finally determines the conditions which must be satisfied by $h_0(r, t)$ across a discontinuity.

We suppose that $t > t_B$ and $h_0(r, t)$ has developed a multiple-valued region. We remove this region by the introduction of a jump discontinuity at $r = s(t)$. Ahead of the discontinuity we let $h_0 = h_A(t)$ while behind $h_0 = h_B(t)$. The situation is illustrated

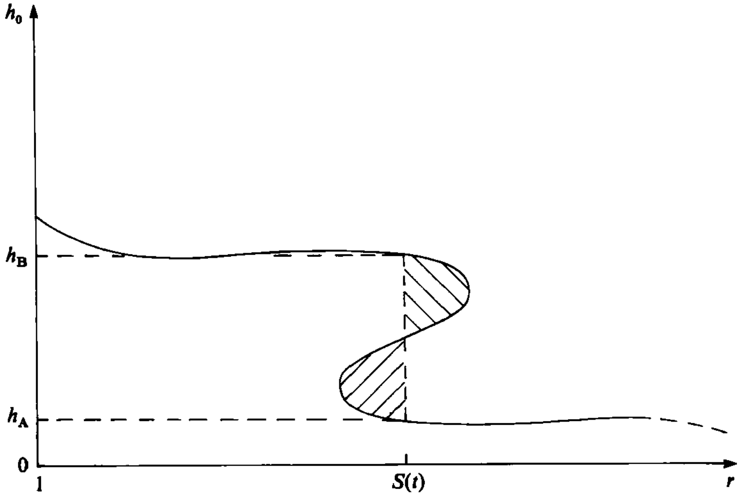


FIGURE 3. A qualitative sketch of the multiple-valued profile of $h_0(r,t)$.

in figure 3, and it is shown in Appendix A that when such a discontinuity is required, then

$$h_B(t) > h_A(t). \tag{57}$$

If the discontinuous solution is to provide a uniform outer approximation to the solution of the full equations when $\epsilon \ll 1$, then we should be able to re-scale the full equations in the vicinity of the discontinuity (so that the highest derivatives in $r$ are retained at leading order) and obtain within this thin region a structure for the discontinuity which matches with $h_0(r,t)$ as we move out of the region on both sides.

To examine the structure region we first introduce the travelling coordinate $y$ defined by $y = r - s(t)$. The discontinuity is now located at $y = 0$. Within this structure region we must have $h$ of $O(1)$ to enable matching with $h_0$ as $y \to \pm\infty$. A balancing of terms in equations (1)–(4) then suggests that the appropriate scaled variables for this region are

$$y = \epsilon\tilde{y}, \quad w = \epsilon^{-1}\tilde{w}, \quad p = \epsilon\tilde{p}, \tag{58a}$$

with $u$ and $r$ remaining of $O(1)$.

We look for solutions in this region as asymptotic expansions in the form

$$h = \tilde{h}_0(\tilde{y},t) + \dots \quad \text{as } \epsilon, Re \to 0, \tag{58b}$$

with similar expansions for $u$, $v$, $\tilde{w}$ and $\tilde{p}$.

Substitution of (58b) into the full equations (when written in terms of the scaled variables) and expanding gives, at leading order,

$$\frac{\partial \tilde{u}_0}{\partial \tilde{y}} + \frac{\partial \tilde{w}_0}{\partial z} = 0, \quad \frac{\partial^2 \tilde{u}_0}{\partial \tilde{y}^2} + \frac{\partial^2 \tilde{u}_0}{\partial z^2} + s(t) - \frac{\partial \tilde{p}_0}{\partial \tilde{y}} = 0, \tag{59a, b}$$

$$\frac{\partial^2 \tilde{v}_0}{\partial \tilde{y}^2} + \frac{\partial^2 \tilde{v}_0}{\partial z^2} - 2u_0 = 0, \quad \frac{\partial^2 \tilde{w}_0}{\partial \tilde{y}^2} + \frac{\partial^2 \tilde{w}_0}{\partial z^2} - \frac{1}{F} - \frac{\partial \tilde{p}_0}{\partial z} = 0. \tag{59c, d}$$

The boundary conditions (5)–(8) become at leading order

$$\left(\tilde{u}_0-\frac{\mathrm{d}s}{\mathrm{d}t}\right)\frac{\partial \tilde{h}_0}{\partial \tilde{y}}-w_0=0, \tag{60a}$$

$$2\left(\frac{\partial \tilde{w}_0}{\partial z}-\frac{\partial \tilde{u}_0}{\partial \tilde{y}}\right)\frac{\partial \tilde{h}_0}{\partial \tilde{y}}+\left(\frac{\partial \tilde{u}_0}{\partial z}+\frac{\partial \tilde{w}_0}{\partial \tilde{y}}\right)\left(1-\left[\frac{\partial \tilde{h}_0}{\partial \tilde{y}}\right]^2\right)=0, \tag{60b}$$

$$\frac{\partial \tilde{v}_0}{\partial z}-\frac{\partial \tilde{v}_0}{\partial \tilde{y}}\frac{\partial \tilde{h}_0}{\partial \tilde{y}}=0, \tag{60c}$$

$$-\tilde{p}_0+2\left[1+\left(\frac{\partial \tilde{h}_0}{\partial \tilde{y}}\right)^2\right]^{-1}\left\{\frac{\partial \tilde{u}_0}{\partial \tilde{y}}\left(\frac{\partial \tilde{h}_0}{\partial \tilde{y}}\right)^2+\frac{\partial \tilde{w}_0}{\partial z}-\frac{\partial \tilde{u}_0}{\partial z}\frac{\partial \tilde{h}_0}{\partial \tilde{y}}-\frac{\partial \tilde{w}_0}{\partial \tilde{y}}\frac{\partial \tilde{h}_0}{\partial \tilde{y}}\right\}-T\frac{\partial^2 \tilde{h}_0}{\partial \tilde{y}^2}\left(1+\left(\frac{\partial \tilde{h}_0}{\partial \tilde{y}}\right)^2\right)^{-\frac{3}{2}}=0, \tag{60d}$$

on $z=\tilde{h}_0(\tilde{y},t)$ for all $\tilde{y}$;

$$u_0=\tilde{w}_0=v_0=0 \quad \text{on } z=0 \text{ for all } \tilde{y}. \tag{60e}$$

The remaining conditions to be satisfied by (59) are the matching conditions. On matching with the outer solution as $\tilde{y}\rightarrow\pm\infty$ we obtain, after using (20),

$$\tilde{h}_0\rightarrow h_i(t), \tag{61a}$$

$$\tilde{u}_0\rightarrow \tfrac{1}{2}s(t)\,z(2h_i(t)-z), \tag{61b}$$

$$\left.\begin{aligned}&\tilde{v}_0\rightarrow \tfrac{1}{12}s(t)\,z[4z^2h_i(t)-z^3-8h^3(t)],\\ &\tilde{w}_0\rightarrow 0,\end{aligned}\right\} \tag{61c}$$

as $\tilde{y}\rightarrow\pm\infty$, where we put $h_i=h_{\mathrm{A}}$ as $\tilde{y}\rightarrow+\infty$, and $h_i=h_{\mathrm{B}}$ as $\tilde{y}\rightarrow-\infty$.

The leading-order structure is now determined by the solution of (59) subject to (60) and (61), which is a quasi-steady, free-surface Stokes problem. This problem is analytically intractable, but again the required relation between $h_{\mathrm{A}}$ and $h_{\mathrm{B}}$ can be obtained without a full solution.

On using (59$a$), (60$a$) and (60$e$) it is easily shown that

$$\int_{z=0}^{z=\tilde{h}_0}\left(\tilde{u}_0-\frac{\mathrm{d}s}{\mathrm{d}t}\right)\mathrm{d}z=\tilde{Q}(t) \quad \text{for all } \tilde{y}. \tag{62}$$

The function $\tilde{Q}(t)$ is determined through conditions (61$a,b$) with $h_i=h_{\mathrm{B}}$ as

$$\tilde{Q}(t)=\tfrac{1}{3}s(t)\,h_{\mathrm{B}}^3(t)-\frac{\mathrm{d}s}{\mathrm{d}t}h_{\mathrm{B}}(t). \tag{63}$$

The relation between $h_{\mathrm{A}}$ and $h_{\mathrm{B}}$ is now obtained by evaluating (62) as $\tilde{y}\rightarrow\infty$ using (61) with $h_i=h_{\mathrm{A}}$ and substituting for $\tilde{Q}$ from (63). After some manipulation we arrive at the condition

$$\frac{\mathrm{d}s}{\mathrm{d}t}=\tfrac{1}{3}(h_{\mathrm{B}}^2(t)+h_{\mathrm{A}}(t)\,h_{\mathrm{B}}(t)+h_{\mathrm{A}}^2(t))\,s, \tag{64}$$

which must be satisfied by the jump discontinuity. Condition (64) together with the characteristic equations (47) or (48) when applied both ahead and behind the discontinuity provide three equations to be solved for $h_{\mathrm{A}}$, $h_{\mathrm{B}}$ and $s$. This fixes uniquely the location and strength of the discontinuity required to replace the multiple-valued region arising in (47) or (48), after which a structure for the discontinuity is found as the solution of (59)–(61). A point to note from (64) is that, since $h_A, h_B, s > 0$, every required discontinuity will move downstream.

### 5.3. *Reformulation as a conservation law*

An interesting reformulation of the outer problem for $h_0$ can be made in noticing that solutions of (34) with multiple-valued regions replaced by discontinuities which satisfy condition (64) are equivalent to single-valued solutions of the following integral conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{r_1}^{r_2} r h_0(r,t)\,\mathrm{d}r\right] + [\tfrac{1}{3}r^2 h_0^3]_{r_1}^{r_2} = 0, \tag{65}$$

for all $r_1, r_2 > 1$. On differentiable sections of the profile $h_0(r,t)$, applying the limit $r_1 \rightarrow r_2$ in (65) results in (34). At points of discontinuity in the profile, the jump conditions obtained from (65) are readily seen to agree with (64).

The reformulation in terms of this conservation law enables us to obtain a constructive method of fitting discontinuities satisfying (64) into multiple-valued sections of (47) or (48). This method is equivalent to the 'equal area' rule of Whitham (1974). We introduce the new coordinate $R = r^2$. In terms of $R$, (65) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{R_1}^{R_2} h_0(R,t)\,\mathrm{d}R\right] + [\tfrac{2}{3}R h_0^3]_{R_1}^{R_2} = 0, \tag{66}$$

and at differentiable points,

$$\frac{\partial h_0}{\partial t} + 2Rh_0^2 \frac{\partial h_0}{\partial R} = -\tfrac{2}{3}h_0^3, \tag{67}$$

When the solution of (67) becomes multiple valued it must be replaced by an appropriate discontinuity. Now, both the multiple-valued and discontinuous solutions must satisfy the conservation law (66), and all solutions of (66) satisfy the following property (see Appendix B),

$$\alpha(t) = \frac{2}{3}\int_0^t [A(0)^3 - A(p)^3]\,\mathrm{d}p + \alpha(0), \tag{68}$$

where
$$\alpha(t) \equiv \int_1^\infty h_0\,\mathrm{d}R,$$

which is the area under the profile of $h_0$. Since both the multiple-valued and discontinuous solutions also have the same initial and boundary conditions, they must both satisfy (68), i.e. the two profiles must enclose the same area. Hence the discontinuity must be inserted into the multiple-valued profile so as to preserve the area under the profile. The reformulation has thus produced a convenient analytical approach to obtaining $h_0(r,t)$: obtain the solution of (67) via the method of characteristics and replace multiple-valued regions by discontinuities which satisfy the equal area rule.

The theory presented in this section can be stated concisely in the following result (in terms of the coordinate $R = r^2$):

*Result* (*ii*) In region I a uniform leading-order approximation in $R > 1$ as $\epsilon \rightarrow 0$ (with $t = O(1)$) is obtained as the solution of (67), with multiple-valued regions replaced by discontinuities satisfying the 'equal area' rule.

We next consider a particular case of the initial-boundary value problem discussed here. This describes the response of the film to a sudden change in the driving flux.

## 6. The response of the film to a flux change

To examine the response of the film to a sudden change in driving flux, we consider the initial-boundary-value problem discussed in the previous section, with

$$A(t) = \begin{cases} A_- & (t = 0) \\ A_+ & (t > 0), \end{cases}$$

$$H(r) = A_- r^{-\frac{2}{3}} \quad (r > 1),$$

where $A_-, A_+$ are positive constants. To decide whether the solution $h_0(r, t)$ remains single-valued we view the jump in $A(t)$ at $t = 0$ as the limiting case of a rapid, but smooth transition in the neighbourhood of $t = 0$. We can then appeal to result (i).

(i) $A_- > A_+$

This case represents a reduction in the driving flux. An application of result (i) shows that $h_0(r, t)$ remains single-valued for all $t > 0$. The solution is readily obtained through (47) and (48) as,

$$h_0(r,t) = \begin{cases} A_+ r^{-\frac{2}{3}} & (1 \leqslant r \leqslant (\frac{4}{3}A_+^2 t + 1)^{\frac{3}{4}}), \\ \dfrac{\sqrt{3}}{2t^{\frac{1}{2}}}(1 - r^{-\frac{4}{3}})^{\frac{1}{2}} & ((\frac{4}{3}A_+^2 t + 1)^{\frac{3}{4}} < r < (\frac{4}{3}A_-^2 t + 1)^{\frac{3}{4}}), \\ A_- r^{-\frac{2}{3}} & (r \geqslant (\frac{4}{3}A_-^2 t + 1)^{\frac{3}{4}}). \end{cases} \tag{69}$$

An examination of (69) shows that the steady profile corresponding to the old flow rate is washed away downstream with speed $A_-^2(\frac{4}{3}A_-^2 t + 1)^{-\frac{1}{4}}$, while the steady profile corresponding to the new flow rate advances downstream with speed $A_+^2(\frac{4}{3}A_+^2 t + 1)^{-\frac{1}{4}}$. They are separated by a spreading transition region which is monotone increasing with $r$.

(ii) $A_+ > A_-$

This case represents an increase in the driving flux of the film. An application of result (i) shows that the profile becomes multiple valued in finite time. In fact $t_B = 0$ and a discontinuity is required immediately at $r = 1$. The appropriate discontinuous solution is readily obtained via result (ii) as

$$h_0(r,t) = \begin{cases} A_+ r^{-\frac{2}{3}} & (1 < r < s(t)), \\ A_- r^{-\frac{2}{3}} & (r > s(t)), \end{cases}$$

with $h_A = A_- s^{-\frac{2}{3}}$, $h_B = A_+ s^{-\frac{2}{3}}$ and the discontinuity position given by

$$s(t) = (\tfrac{4}{9}[A_+^2 + A_+ A_- + A_-^2]\, t + 1)^{\frac{3}{4}}.$$

This completes the solution. Thus there is a discontinuous transition between the steady states at the old and new flow rates. This propagates downstream at a speed

$$\dot{s}(t) = \tfrac{1}{3}(A_+^2 + A_+ A_- + A_-^2)\,[\tfrac{4}{9}(A_+^2 + A_+ A_- + A_-^2)\, t + 1]^{-\frac{1}{4}}.$$

The strength of the discontinuity is given by

$$h_B - h_A = (A_+ - A_-)(\tfrac{4}{9}[A_+^2 + A_+ A_- + A_-^2]\, t + 1)^{-\frac{1}{2}}.$$

## 7. Instability of the film

The theory of the previous sections has provided in region I a leading-order approximation as $\epsilon \to 0$, which is uniform in $r > 1$ when $t$ is $O(1)$. The only further source of non-uniformity may occur as $t \to \infty$. To assess whether or not the theory of §6 remains uniform for $t \gg 1$ would require obtaining higher-order terms in expansions (31) and comparing their behaviour as $t \to \infty$ with that of the leading-order terms. Unfortunately, the complexity of the leading-order terms makes this approach intractable, except in the case of small-amplitude disturbances to the steady film. We can take advantage of this to obtain an estimate of the uniformity as $t \to \infty$, since, according to the leading-order theory the amplitude of the disturbances diminishes as $t \to \infty$, and it is reasonable to expect that the final, old-age behaviour will be described accurately by a linearized theory.

We consider the development in I of a small-amplitude disturbance to the steady film $h_s(r) = Ar^{-\frac{2}{3}}$. The initial and boundary conditions are then,

$$h(r,0) = h_s(r) + \beta\phi(r) \quad (r > 1), \tag{70}$$

$$h(1,t) = A(=[3\sigma]^{\frac{1}{3}}) \quad (t > 0), \tag{71}$$

where $|\beta| \ll \epsilon$ is a measure of the amplitude of the disturbance and $\phi(r)$ is a bounded function. A solution to the full equations (1)–(4) is sought in the form

$$h = h_s(r) + \beta\bar{h}(r,t;\epsilon), \tag{72}$$

together with similar expressions for $u$, $v$, $w$ and $p$,

$$\left.\begin{aligned} u &= u_s(r,z) + \beta(\bar{u}_0 + \epsilon\bar{u}_1 + \dots), \\ v &= v_s(r,z) + \beta(\bar{v}_0 + \epsilon\bar{v}_1 + \dots), \\ w &= w_s(r,z) + \beta(\bar{w}_0 + \epsilon\bar{w}_1 + \dots), \\ p &= p_s(r,z) + \beta(\bar{p}_0 + \epsilon\bar{p}_1 + \dots), \end{aligned}\right\} \tag{73}$$

where $u_s$, $v_s$, $w_s$ and $p_s$ are the steady-state forms of $u$, $v$, $w$ and $p$ respectively. The first two terms of expansions (73) are readily obtained after substitution into equations (1)–(4) and boundary conditions (6)–(8). Finally the substitution into (5) for $u$, $w$ and $h$ from (72) and (73) leads to the following linear equation for $\bar{h}$,

$$\frac{\partial \bar{h}}{\partial t} + \frac{A^2}{r^{\frac{1}{3}}}\frac{\partial \bar{h}}{\partial r} + \frac{2}{3}\frac{A^2}{r^{\frac{4}{3}}}\bar{h} - \frac{\epsilon B}{r}\frac{\partial}{\partial r}\left(\frac{\bar{h}}{r^2}\right) = \epsilon\frac{C}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \bar{h}}{\partial r}\right) + O(\epsilon^2, \beta), \tag{74}$$

where $B = (\frac{509}{1296})A^6 Re - (\frac{2}{3}F)A^3$ and $C = A(\frac{2}{3}F)(1 - \frac{2}{5}A^3 ReF)$. The first three terms on the left-hand side of (74) are the linearization of (34), with the remaining terms giving the correction up to $O(\epsilon)$. In terms of $\bar{h}$, conditions (70) and (71) become

$$\bar{h}(r,0) = \phi(r) \quad (r > 1), \tag{75a}$$

$$\bar{h}(1,t) = 0 \quad (t > 0). \tag{75b}$$

Equation (74) is simplified on making the transformation

$$\eta = r^{\frac{2}{3}}\bar{h}, \quad \xi = r^{\frac{4}{3}} - C_0 t, \quad \tau = \log t, \tag{76}$$

where $C_0 = \frac{4}{3}A^2$. In terms of $\xi$, $\tau$ and $\eta$, (74) becomes

$$\frac{\partial \eta}{\partial \tau} = \tfrac{4}{3}\epsilon B \exp(-\tau)\frac{\partial}{\partial \xi}\left[\frac{\eta}{(\xi \exp(-\tau)+C_0)^2}\right] + \tfrac{16}{9}\epsilon C \frac{\partial}{\partial \xi}\left[\frac{1}{(\xi \exp(-\tau)+C_0)^{\frac{1}{2}}}\frac{\partial}{\partial \xi}\left\{\frac{\eta}{(\xi \exp(-\tau)+C_0)^{\frac{1}{2}}}\right\}\right] + O(\epsilon^2, \beta). \quad (77)$$

The domain of solution is transformed to $-\infty < \tau < \infty$ with $1 - C_0 \exp(\tau) < \xi < \infty$, and conditions (75) become

$$\eta \to \xi^{\frac{1}{2}}\hat{\phi}(\xi) \quad \text{as } \tau \to -\infty, \quad \xi > 1 \quad (78a)$$

$$\eta = 0 \quad \text{at } \xi = 1 - C_0 \exp(\tau), \quad (78b)$$

where
$$\hat{\phi}(\xi) \equiv \phi(\xi^{\frac{3}{4}}).$$

We look for a solution of (77) in the form

$$\eta(\xi, \tau; \epsilon) = \eta_0 + \epsilon \eta_1 + \dots . \quad (79)$$

On substituting for $\eta$ from (79) into (77) and (78), expanding and equating like power of $\epsilon$, we obtain a hierarchy of problems to be solved in turn. The first two problems are readily solved, to give

$$\left.\begin{aligned} \eta_0(\xi,\tau) &= \begin{cases} \xi^{\frac{1}{2}}\hat{\phi}(\xi) & (\xi > 1), \\ 0 & (1 - C_0\exp(\tau) < \xi \leqslant 1), \end{cases} \\ \eta_1(\xi,\tau) &= q_1(\xi,\tau) + q_2(\xi,\tau), \end{aligned}\right\} \quad (80)$$

where

$$\left.\begin{aligned} q_1(\xi,\tau) &= \begin{cases} \dfrac{4}{3}\dfrac{(B-2C)(\xi^{\frac{1}{2}}\hat{\phi}(\xi))'}{\xi(\xi\exp[-\tau]+C_0)^2} - \dfrac{8}{9}\dfrac{(3B-2C)\hat{\phi}(\xi)(C_0 - \xi(\xi\,\mathrm{e}^{-\tau}+C_0))}{\xi^{\frac{5}{2}}(\xi\exp[-\tau]+C_0)^2} & (\xi > 1), \\ 0 \quad (1 - C_0\exp(\tau) < \xi \leqslant 1). \end{cases} \\ q_2(\xi,\tau) &= \begin{cases} \dfrac{16}{9}\dfrac{C}{C_0}\dfrac{\mathrm{d}^2(\xi^{\frac{1}{2}}\hat{\phi}(\xi))}{\mathrm{d}\xi^2}\log\left[1 + \dfrac{C_0}{\xi}\exp(\tau)\right] \quad (\xi > 1), \\ 0 \quad (1 - C_0\exp(\tau) < \xi \leqslant 1). \end{cases} \end{aligned}\right\} \quad (81)$$

Examining (80) and (81) it is seen that $\eta_0$ and $q_1$ remain bounded for all $\xi$ and $\tau$, but in $\xi > 1$,

$$q_2(\xi,\tau) \sim \frac{16}{9}\frac{C}{C_0}\frac{\mathrm{d}^2(\xi^{\frac{1}{2}}\hat{\phi}(\xi))\tau}{\mathrm{d}\xi^2} \quad \text{as } \tau \to \infty.$$

This renders expansion (79) non-uniform when $\tau = O(\epsilon^{-1})$. To obtain a uniform expansion when $\tau = O(\epsilon^{-1})$ we introduce an outer region with the scaled time variable,

$$\tau^* = \epsilon\tau. \quad (82)$$

In this region (79), (80) and (81) suggest $\eta$ is $O(1)$. In terms of $\tau^*$, (77) becomes

$$\frac{\partial \eta}{\partial \tau^*} = \tfrac{4}{3}B \exp\left(-\frac{\tau^*}{\epsilon}\right)\frac{\partial}{\partial \xi}\left[\frac{\eta}{(\xi \exp(-\tau^*/\epsilon)+C_0)^2}\right] + \tfrac{16}{9}C \frac{\partial}{\partial \xi}\left[\frac{1}{(\xi \exp(-\tau^*/\epsilon)+C_0)^{\frac{1}{2}}}\frac{\partial}{\partial \xi}\left\{\frac{\eta}{(\xi \exp(-\tau^*/\epsilon)+C_0)^{\frac{1}{2}}}\right\}\right]. \quad (83)$$

An examination of (83) shows that within this region we have,

$$\eta(\xi,\tau^*,\epsilon) \sim \eta^*(\xi,\tau^*)+\{\text{terms exponentially small in } \epsilon\}, \tag{84}$$

where $\eta^*$ satisfies the linear diffusion equation,

$$\frac{\partial\eta^*}{\partial\tau^*} = \kappa\frac{\partial^2\eta^*}{\partial\xi^2} \quad (-\infty<\xi<\infty), \tag{85}$$

with $\kappa = 16C/9C_0$. A formal solution of (85) is given by

$$\eta^*(\xi,\tau^*) = \int_{-\infty}^{\infty} \mathrm{D}(k)\,\mathrm{e}^{-\kappa k^2\tau^*}\,\mathrm{e}^{\mathrm{i}k\xi}\,\mathrm{d}k \quad (-\infty<\xi<\infty). \tag{86}$$

The asymptotic matching of expansions (79) (as $\tau\to\infty$) and (84) (as $\tau^*\to 0$) then determines $\mathrm{D}(k)$ as

$$\mathrm{D}(k) = \frac{1}{2\pi}\int_1^{\infty} s^{\frac{1}{2}}\hat{\phi}(s)\,\mathrm{e}^{-\mathrm{i}ks}\,\mathrm{d}s. \tag{87}$$

The long-time asymptotics of the film are thus given by (86).

The analysis of this section has indicated that a non-uniformity in the theory of §4 will arise when $t \gg 1$ (with a possible estimate, using (82) and (76), being $t$ of $O(\mathrm{e}^{1/\epsilon})$, after which the old-age development of the film is governed by the linear diffusion equation (85) with diffusion coefficient

$$\kappa = \frac{4A}{9F}\left(1-\tfrac{2}{5}A^3ReF\right).$$

For $Re \ll 1$, then $\kappa > 0$, and the disturbance ultimately decays according to (86). In this case, the final old-age behaviour is of little interest, with the main development determined by the theory of §7.

However, when $Re$ is $O(1)$, $\kappa$ may become negative, with the solution (86) becoming unbounded. This indicates instability of the steady film. In this case the film will ultimately diverge from the steady state and possibly develop into a quasi-steady 'wavy' form. On defining a modified Reynolds number $\overline{Re} = Q\Omega^2 l/g\nu$ $(= \Omega^2 Q_T/2\pi g\nu)$, the condition for instability of the film, $\kappa < 0$, may be written as

$$\overline{Re} > \tfrac{5}{6}.$$

This condition is similar to that obtained by Benjamin (1957) determining the stability of a thin liquid film flowing down an inclined plane.

It should be noted here that when the condition for instability is satisfied, the growth rate as determined by (85) becomes unbounded with increasing wavenumber. This physically unacceptable result arises since surface-tension effects have been neglected due to the long-wave approximation. However, for large wavenumbers (small wavelengths) surface-tension effects will be significant and we expect the inclusion of such terms will restore a bounded growth rate. The same problem was encountered by Benney (1966) when studying the linearized stability of thin film flows down an inclined plane. Gjevik (1970) demonstrated in this case that the effect of surface tension does indeed recover a finite growth rate.

## 8. Discussion

We have considered the axisymmetric thin liquid film formed on a horizontally spinning disk. For $\epsilon \ll 1$, the asymptotic structure of both the steady and unsteady behaviour of the film has been obtained. The steady film has a thin inner region close to the inlet within which rapid adjustment to the inlet conditions takes place. In the main outer region the film thickness is monotone decreasing with distance from the inlet.

In the unsteady case we have examined the response of the steady film to localized disturbances and/or changes in the driving flux. Here we again required a thin 'inlet' region and also a region for $t \ll 1$ in which rapid adjustment to initial conditions occurs. Through matching, these regions provide appropriate 'boundary' and 'initial' conditions for the leading-order problem in the main region. In the main region we found that it is possible for multiple-valued solutions to develop, and the conditions under which this occurs were derived. Such multiple-valued regions will not occur in the solution of the full equations, and we interpret this as arising owing to the appearance of a local non-uniformity in the leading-order approximations. In the spirit of Crighton & Scott (1979), it is demonstrated via the method of matched asymptotic expansions, that a uniform approximation is recovered by replacing the multiple-valued region by an appropriate jump discontinuity. The discontinuities satisfy an 'equal area' rule of the type discussed by Whitham (1974).

Finally we considered whether or not the approximations in the main region remain uniform for $t \gg 1$. We attempted to assess this by considering the evolution of a small-amplitude disturbance imposed upon the steady film, for which we can obtain the first correction to the leading-order term. This demonstrated that a non-uniformity does occur for $t \gg 1$. When $Re \ll 1$ the old-age behaviour is dominated by diffusion. However, for $Re$ of $O(1)$ it is possible for the film to become unstable. Under these conditions we expect that the film will eventually develop into a quasi-steady 'wavy' form.

To conclude we note that the restriction to axial symmetry will in general be a physically realistic assumption, since most thin film flows generated on rotating disks are formed by an axisymmetric mechanism. Thus changes to the flow rate generating the film will be uniform in the angular direction, producing essentially axisymmetric disturbances to the film. However, this may not be the case when the film is unstable; here non-axisymmetric disturbances may grow spontaneously.

## Appendix A

Here we show that when a discontinuity is required to remove a multiple-valued region, then the discontinuity necessarily satisfies condition (57). Suppose a discontinuity is inserted at $r = s(t)$, with $h_0 = h_A(t)$ ahead and $h_0 = h_B(t)$ behind, as shown in figure 3. We now revert to the $(x, t)$ characteristic plane. In this plane the discontinuity is located at $P_1(\log s(t), t)$. By hypothesis, this must be a point of characteristic intersection, and on one of the characteristics $h_0 = h_A$, while on the other $h_0 = h_B$.

Suppose both of these characteristics are from the family (47*b*), and let $\xi = \xi_A, \xi_B$ on the characteristic curves for which $h_0 = h_A, h_B$ at $P$, respectively. Since the discontinuity must remove the multiple-valued region then we can deduce that

$$\xi_A > \xi_B. \tag{A 1}$$

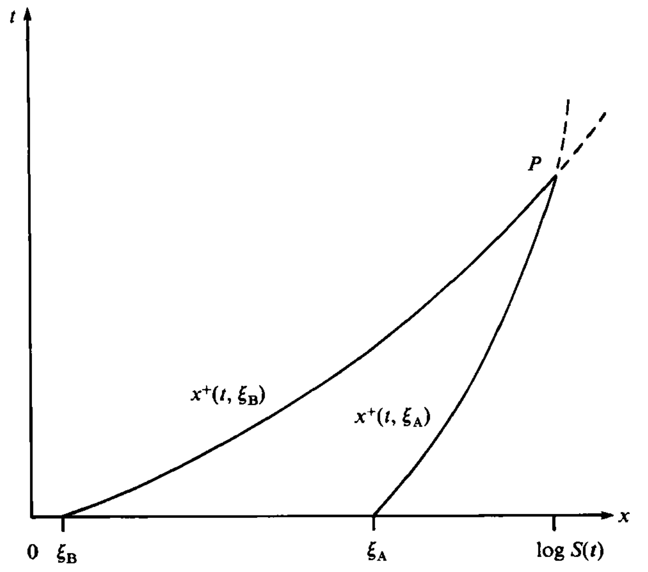


FIGURE 4. A sketch in the $(x, t)$-plane of the intersecting characteristics $x^+(t, \xi_A)$ and $x^+(t, \xi_B)$.

Furthermore, from (47b), the characteristics are monotone increasing functions of $t$, which together with (A 1) and the fact that intersection occurs at $P$ leads to the inequality

$$\left[\frac{dx^+}{dt}(t, \xi_B) - \frac{dx^+}{dt}(t, \xi_A)\right]_P > 0. \tag{A 2}$$

On evaluating the derivatives through (47b) and using (47a) we can re-write (A 2) as

$$\psi(t, \xi_B) - \psi(t, \xi_A) > 0,$$

which, via (42), becomes

$$h_B > h_A. \tag{A 3}$$

For the two further cases in which the two characteristics are both from the family (48b) or one from (47b) and the other from (48b), the result (A 3) follows in the same way.

A geometric representation of inequality (A 2) in the $(x, t)$-plane is shown in figure 4.

## Appendix B

For localized initial disturbances, we have, from (42) and (44),

$$h_0(1, t) = A(t), \quad h_0(R, t) \sim A(0)\, R^{-\frac{1}{3}} \quad \text{as } R \to \infty. \tag{B 1}$$

On letting $R_2 \to \infty$ and $R_1 \to 1$ in (66) we obtain, after use of (B 1),

$$\frac{d\alpha}{dt}(t) + \tfrac{2}{3}(A(0)^3 - A(t)^3) = 0.$$

An integration of this expression leads directly to (68).

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